MOTION OF A RIGID STAMP ON THE BOUNDARY OF A VISCOELASTIC HALF-PLANE

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The plane contact problem of motion of a rigid stamp at constant velocity over the boundary of a half-plane is investigated. The material filling the medium is assumed isotropic and linearly viscoelastic. Such velocities of motion are considered for which it is impossible to neglect the influence of inertial forces. A numerical example is presented.

1. A number of contact problems for linear viscoelastic bodies has been investigated in [1 to 3], however, the influence of inertial forces was neglected. If the rate of stamp motion is of the order of the velocity of sound, then the influence of the inertial forces will be significant. The problem of a stamp moving over the boundary of an elastic half-plane has been considered by one of the authors [4]. It should be noted that taking account of viscoelastic effects and inertial forces results in substantial complications in solving the problem.

We find an expression for the normal component of the displacement on the surface of a viscoelastic half-plane subjected to a concentrated force moving along it with the constant velocity w. We henceforth consider the concentrated force as the limiting case of pressure distributed in some interval.

As it turns out, particularly in [5], for the majority of viscoelastic bodies it can be assumed that the volume strain is purely elastic, and volume aftereffect can hence be neglected. Then the relationships between the strain and stress components for the state of plane strain will be

$$\sigma_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_x - \int_{-\infty}^{t} R(t-\tau)\,\varepsilon_x\,d\tau + \frac{\nu}{1-\nu}\varepsilon_y + \frac{1}{2}\int_{-\infty}^{t} R(t-\tau)\,\varepsilon_y\,d\tau\right]$$

$$\sigma_y = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{\nu}{1-\nu}\varepsilon_x + \frac{1}{2}\int_{-\infty}^{t} R(t-\tau)\,\varepsilon_x\,d\tau + \varepsilon_y - \int_{-\infty}^{t} R(t-\tau)\,\varepsilon_y\,d\tau\right]$$

$$\tau_{xy} = \frac{E}{1+\nu}\left[\gamma_{xy} - \frac{3}{2}\,\frac{1-\nu}{1-2\nu}\int_{-\infty}^{t} R(t-\tau)\,\gamma_{xy}\,d\tau\right] \qquad (1.1)$$

Henceforth, the case when the kernel in the expressions in (1.1) is exponential

$$R(t-\tau) = m e^{-n(t-\tau)/t_0}, \qquad m>0, \qquad n>0 \qquad (1.2)$$

will be considered.

Let us introduce some characteristic time t_0, and let us consider media for which the aftereffect is not very significant, and therefore, the parameter $n_1 = n/t_0$ is large. Let us utilize Eqs.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2} \tag{1.3}$$

and the relationships

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{1.4}$$

The boundary conditions of the problem under consideration are

$$\tau_{xy} = 0, \quad \sigma_y = -P\delta(x) \quad \text{for} \quad y = 0, \ -\infty < x < \infty \tag{1.5}$$

$$\sigma_x, \sigma_y, \tau_{xy} \to 0 \quad \text{for} \quad (x^2 + y^2)^{1/2} \to \infty \tag{1.6}$$

Here P is the magnitude of the concentrated forces, and $\delta(x)$ is the delta function.

2. We shall seek an elementary solution in the form

$$\begin{gathered}\sigma_x = e^{i\beta x}\psi_1(\beta, y), \qquad \varepsilon_x = e^{i\beta x}\varphi_1(\beta, y) \\ \sigma_y = e^{i\beta x}\psi_2(\beta, y), \quad \varepsilon_y = e^{i\beta x}\varphi_2(\beta, y), \quad u = e^{i\beta x}\omega_1(\beta, y) \\ \tau_{xy} = e^{i\beta x}\psi_3(\beta, y), \quad \gamma_{xy} = e^{i\beta x}\varphi_3(\beta, y), \quad v = e^{i\beta x}\omega_2(\beta, y)\end{gathered} \tag{2.1}$$

We shall take the real parts of the expressions obtained in order to find the stress, strain and displacement components.

Substituting (2.1) into (1.1), (1.3), (1.4), introducing dimensionless functions and coordinates, and transferring to a moving coordinate system, we obtain

$$\begin{gathered}\psi_1(\beta, y) = A\varphi_1(\beta, y) + B\varphi_2(\beta, y), \qquad \psi_2(\beta, y) = B\varphi_1(\beta, y) + A\varphi_2(\beta, y) \\ \psi_3(\beta, y) = C\varphi_3(\beta, y), \qquad i\beta\psi_1(\beta, y) + \psi_3'(\beta, y) + N\beta^2\omega_1(\beta, y) = 0 \\ \psi_2'(\beta, y) + i\beta\psi_3(\beta, y) + N\beta^2\omega_2(\beta, y) = 0, \qquad \varphi_1(\beta, y) = i\beta\omega_1(\beta, y) \\ \varphi_2(\beta, y) = \omega_2'(\beta, y), \qquad \varphi_3(\beta, y) = {}^1/_2\,[\omega_1'(\beta, y) + i\beta\omega_2(\beta, y)]\end{gathered} \tag{2.2}$$

Here

$$A = \frac{1-\nu}{(1+\nu)(1-2\nu)}(1 - m_1H), \qquad B = \frac{\nu}{(1+\nu)(1-2\nu)}\left(1 + \frac{1-\nu}{\nu}\frac{m_1}{2}H\right)$$

$$C = \frac{1}{1+\nu}\left(1 - \frac{3}{2}\frac{1-\nu}{1-2\nu}m_1H\right), \quad H = [n_1x_0/w + i\beta]^{-1}, \quad N = \frac{\rho w^2}{E}, \quad m_1 = \frac{mx_0}{w}$$

where x_0 is some provisional linear scale.

The system (2.2) can be reduced to one ordinary differential Eq.

$$C_1\psi_2^{\mathrm{IV}}(\beta, y) + C_2\psi_2''(\beta, y) + C_3\psi_2(\beta, y) = 0 \tag{2.3}$$

Here

$$C_1 = AC, \quad C_2 = \beta^2(2BC + 2B^2 - 2A^2 + NC + 2NA),$$
$$C_3 = \beta^4(AC - NC - 2NA + 2N^2)$$

Henceforth, all the functions in the relationships (2.2) will be expressed in terms of $\psi_2(\beta, y)$.

Taking into account the constraints imposed earlier on the kernel (1.2), we transform the coefficients of (2.3) by expanding them in terms of H, and neglecting terms containing H in powers higher than the first. The solution of (2.3) is

$$\psi_2(\beta, y) = \sum_{k=1}^{4} Q_k e^{\lambda_k y} \tag{2.4}$$

Here the λ_k are roots of the characteristic equation, all distinct, and determined by the relationships

$$\lambda_1 = -\lambda_2 = \beta a_1 (1 + b_1 H), \qquad a_{1,2} = \left[\frac{\pm (L_2^2 - 4L_4)^{1/2} - L_2}{2}\right]^{1/2} \tag{2.5}$$
$$\lambda_3 = -\lambda_4 = \beta a_2 (1 + b_2 H)$$
$$b_{1,2} = \frac{1}{2}\left[\frac{(L_2 L_3 - 2L_1 L_4 - 2L_5)(L_2^2 - 4L_4)^{-1/2} \mp L_3}{(L_2^2 - 4L_4)^{1/2} \mp L_2} - L_1\right]$$

where

$$L_1 = -m_1 \frac{5 - 7\nu}{2(1 - 2\nu)}, \qquad L_4 = 1 - N\frac{(1+\nu)(3 - 4\nu)}{1 - \nu} + 2N^2 \frac{(1+\nu)^2 (1 - 2\nu)}{1 - \nu}$$
$$L_2 = -2 + N\frac{(1+\nu)(3 - 4\nu)}{1 - \nu}, \qquad L_3 = m_1\left[\frac{5 - 7\nu}{1 - 2\nu} - \frac{7}{2} N(1 + \nu)\right]$$
$$L_5 = m_1\left[-\frac{(5 - 7\nu)(1 + \nu)^2}{2(1 - \nu)} + \frac{7}{2} N(1 + \nu)\right]$$

The first subscripts in the expressions for a and b correspond to the choice of the upper sign, and the second subscripts to the lower sign.

Utilizing the condition (1.6) at infinity, we arrive at the conclusion that only $\lambda_k > 0$ should enter into the solution. An analysis of (2.5) permits the conclusion that under the assumptions made above the positive roots are λ_1 and λ_3. On the basis of (2.4) we obtain

$$\psi_2(\beta, y) = Q_1 e^{-\beta a_1 y}(1 - \beta a_1 b_1 H y) + Q_2 e^{-\beta a_2 y}(1 - \beta a_2 b_2 H y), \quad y > 0 \tag{2.6}$$

We seek the expression for σ_y in the following form:

$$\sigma_y = \operatorname{Re}\int_0^\infty \psi_2(\beta, y) e^{i\beta x} d\beta$$

The coefficients Q_j ($j = 1, 2$) are found in such a manner that the boundary conditions (1.5) would be satisfied. Utilizing the second of them, we obtain

$$Q_1 + Q_2 = -\frac{P}{\pi E} \tag{2.7}$$

We establish an expression for $\psi_3(\beta, y)$ from the system (2.2):

$$\psi_3(\beta, y) = iC_4\left[\frac{C_5}{\beta}\psi_2'(\beta, y) + \frac{C_6}{\beta^3}\psi_2'''(\beta, y)\right] \tag{2.8}$$

where we have introduced the notation

$$C_4 = C\,[(C - 2N)(A + B)(A - B - N)]^{-1}$$
$$C_5 = A^2 - B^2 - 2NB - NA - N^2, \qquad C_6 = -NA$$

The expression for τ_{xy} will be

$$\tau_{xy} = \operatorname{Re} \int_0^\infty \psi_3(\beta, y) e^{i\beta x} d\beta$$

Taking account of the relationship (2.6), we transform (2.8) to

$$\psi_3(\beta, y) = -i \{Q_1 e^{-\beta a_1 y} [r_1 + H(r_3 - r_5 \beta y)] + \\ + Q_2 e^{-\beta a_2 y} [r_2 + H(r_4 - r_6 \beta y]\}, \quad y > 0 \tag{2.9}$$

Here $r_1, \ldots, r_6$ are constants independent of β and expressed thus

$$r_{1,2} = a_{1,2} L_6 (L_8 + a_{1,2}^2 L_{10}), \qquad r_{5,6} = b_{1,2} r_{1,2}$$

$$r_{3,4} = L_7 (L_8 + a_{1,2}^2 L_{10}) + L_6 (L_9 + b_{1,2} L_8) + a_{1,2}^2 L_6 (L_{11} + 3b_{1,2} L_{10})$$

$$L_6 = \frac{(1+\nu)^2 (1-2\nu)}{1 - 3N(1+\nu) + 2N^2(1+\nu)^2}, \qquad L_8 = \frac{1}{(1+\nu)^2(1-2\nu)} - N \frac{1}{1-2\nu} - N^2$$

$$L_7 = m_1 \frac{(1+\nu)^2 (1-\nu) [2(2-\nu) - 3N(1+\nu)(1-2\nu) - 4N^2(1+\nu)^3]}{2[1 - 3N(1+\nu) + 2N^2(1+\nu)]^2]^2}$$

$$L_9 = -m_1 \frac{2 - 3\nu + \nu^2}{(1+\nu)^2(1-2\nu)^2}, \quad L_{10} = -N \frac{1-\nu}{(1+\nu)(1-2\nu)}, \quad L_{11} = -m_1 L_{10}$$

It should be noted that in selecting the first subscript in the relationships for r_i ($i = 1, \ldots, 6$), it is necessary to utilize expressions for its components taken also with the first subscripts.

Using the first condition in (1.5), we obtain the missing relationship to determine the coefficients Q_j ($j = 1, 2$)

$$Q_1 (r_1 + Hr_3) + Q_2 (r_2 + Hr_4) = 0 \tag{2.10}$$

The system of Eqs. (2.7) and (2.10) yields

$$Q_{1,2} = \pm \frac{P}{\pi E} \frac{1}{r_1 - r_2} \left(r_{2,1} + H \frac{r_1 r_4 - r_2 r_3}{r_1 - r_2} \right) \tag{2.11}$$

Utilizing the system (2.2), let us consider the relationship governing the displacement of points of a viscoelastic half-plane in the y-direction.

$$\omega_2(\beta, y) = -\frac{1}{N\beta^2} [\psi_2'(\beta, y) + i\beta\psi_3(\beta, y)] \tag{2.12}$$

Then

$$v = \operatorname{Re} \int_0^\infty \omega_2(\beta, y) e^{i\beta x} d\beta$$

Let us transform this relationship by taking account of (2.6), (2.9) and (2.11)

$$v(x, y) = \int_0^\infty [S_1(\beta, y) \cos \beta x + S_2(\beta, y) \sin \beta x] d\beta \tag{2.13}$$

Here

$$S_1(\beta, y) = \frac{P}{\pi E} \frac{l_0}{\beta} \left\{ e^{-\beta a_1 y} \left[l_1 + \frac{\xi_0}{\xi_0^2 + \beta^2} (l_2 + l_3 \beta y) \right] - \\ - e^{-\beta a_2 y} \left[l_4 + \frac{\xi_0}{\xi_0^2 + \beta^2} (l_5 + l_6 \beta y) \right] \right\}$$

$$S_2(\beta, y) = \frac{P}{\pi E} l_0 \left\{ e^{-\beta a_1 y} \frac{1}{\xi_0^2 + \beta^2} (l_2 + l_3 \beta y) - e^{-\beta a_2 y} \frac{1}{\xi_0^2 + \beta^2} (l_5 + l_6 \beta y) \right\}$$

where l_0, ..., l_6 and ξ_0 are constants independent of β and having the form:

$$l_0 = [N(r_1 - r_2)]^{-1}, \qquad l_{2,5} = \frac{(a_{1,2} - r_{1,2})(r_1 r_4 - r_2 r_3)}{r_1 - r_2} + r_{2,1}(a_{1,2} b_{1,2} - r_{3,4})$$

$$l_{1,4} = r_{2,1}(a_{1,2} - r_{1,2}), \qquad l_{3,6} = r_{2,1}(r_{5,6} - a_{1,2}^2 b_{1,2}), \qquad \xi_0 = n_1 x_0 / w$$

(the choice of subscripts is defined above).

Henceforth, we shall deal not with v itself, but with its derivative with respect to x; since it is necessary to find the solution of the integral Eq.

$$\int_{-1}^{1} K(x - \xi) P(\xi) d\xi = f(\xi)$$

where the kernel $K(x)$ will be the Green's function of the problem considered above, and equals $[dv/dx]_{y=0}$, in order to find the stress originating under the stamp. Hence

$$\frac{dv}{dx} = \frac{P}{\pi E} l_0 \left[- \int_0^\infty (l_1 e^{-\beta a_1 y} - l_4 e^{-\beta a_2 y}) \sin \beta x \, d\beta - \right. \tag{2.14}$$

$$- \xi_0 \int_0^\infty (l_2 e^{-\beta a_1 y} - l_5 e^{-\beta a_2 y}) \frac{\sin \beta x}{\xi_0^2 + \beta^2} d\beta - \xi_0 y \int_0^\infty (l_3 e^{-\beta a_1 y} - l_6 e^{-\beta a_2 y}) \beta \frac{\sin \beta x}{\xi_0^2 + \beta^2} d\beta +$$

$$\left. + \int_0^\infty (l_2 e^{-\beta a_1 y} - l_5 e^{-\beta a_2 y}) \beta \frac{\cos \beta x}{\xi_0^2 + \beta^2} d\beta + y \int_0^\infty (l_3 e^{-\beta a_1 y} - l_6 e^{-\beta a_2 y}) \beta^2 \frac{\cos \beta x}{\xi_0^2 + \beta^2} d\beta \right]$$

Let us make a more detailed investigation of the integrals in (2.14). It is easy to show that the first integral in this expression is:

$$\int_0^\infty e^{-a\beta y} \sin \beta x \, d\beta = \frac{x}{x^2 + (ay)^2}$$

while evaluation of the second integral yields

$$\xi_0 \int_0^\infty e^{-a\beta y} \frac{\sin \beta x}{\xi_0^2 + \beta^2} d\beta = \frac{1}{2} \left[e^{-\xi_0 x} \int_{-\infty}^{x} \frac{x e^{\xi_0 x}}{x^2 + (ay)^2} dx - e^{\xi_0 x} \int_{-\infty}^{x} \frac{x e^{-\xi_0 x}}{x^2 + (ay)^2} dx \right] \tag{2.15}$$

It is easy to show boundedness of the third integral in (2.14). Remarking that

$$\int_0^\infty e^{-a\beta y} \frac{\beta \cos \beta x}{\xi_0^2 + \beta^2} d\beta = \frac{d}{dx} \int_0^\infty e^{-a\beta y} \frac{\sin \beta x}{\xi_0^2 + \beta^2} d\beta$$

and utilizing the relationship (2.15), we obtain for the fourth integral

$$\int_0^\infty e^{-a\beta y} \frac{\beta \cos \beta x}{\xi_0^2 + \beta^2} d\beta = - \frac{1}{2} \left[e^{-\xi_0 x} \int_{-\infty}^{x} \frac{x e^{\xi_0 x}}{x^2 + (ay)^2} dx + e^{\xi_0 x} \int_{-\infty}^{x} \frac{x e^{-\xi_0 x}}{x^2 + (ay)^2} dx \right]$$

Finally, the fifth and last integral can be represented as

$$\int_0^\infty e^{-a\beta y} \frac{\beta^2 \cos \beta x}{\xi_0^2+\beta^2}\, d\beta = \int_0^\infty e^{-a\beta y} \cos \beta x \, d\beta - \xi_0^2 \int_0^\infty e^{-a\beta y} \frac{\cos \beta x}{\xi_0^2+\beta^2}\, d\beta$$

The first integral herein is

$$\int_0^\infty e^{-a\beta y} \cos \beta x \, d\beta = \frac{ay}{x^2+(ay)^2}$$

It is not difficult to prove the boundedness of the second integral in this relationship.

Performing a passage to the limit, and taking account of the boundedness of the integrals which are coefficients of y, we finally obtain

$$\frac{dv(x)}{dx} = \lim_{y\to 0} \frac{dv(x, y)}{dx} = \frac{P}{\pi E} l_0 \left[(l_4 - l_1) \frac{1}{x} + (l_5 - l_2) e^{-\xi_0 x} Ei(\xi_0 x) \right] \quad (2.16)$$

Using the asymptotic representation of exponential integral functions $Ei(\xi_0 x)$ and expanding $\exp(-\xi_0 x)$ in power series, we obtain the following approximate expression for the kernel

$$K(x) = \frac{dv(x)}{dx} = \frac{P}{\pi E} l_0 \Big\{ (l_4 - l_1) \frac{1}{x} + (l_5 - l_2) [\ln x + x \ln x +$$
$$+ (C^\circ + \ln \xi_0) + x(C^\circ + \ln \xi_0 + \xi_0) + o(x)] \Big\} \quad (2.17)$$

where C° is the Euler constant.

3. The problem of determining the pressure which originates under a rigid stamp moving at the constant velocity w over the boundary of a viscoelastic half-plane (Fig. 1) can be reduced to solving some singular integral equation.

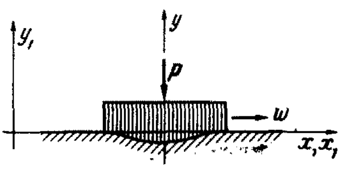

Fig. 1

Let us assume that the dimensions of the contact area are known, and there are no friction forces between the stamp and the viscoelastic half-plane. Then

$$f(x) = \int_{-1}^{1} P(\xi) \left[\frac{A_0}{x-\xi} + A_1 \ln|x-\xi| + A_2 + \right.$$
$$\left. + A_3(x-\xi) + A_1(x-\xi) \ln|x-\xi| + \cdots \right] d\xi \quad (3.1)$$

Here $P(\xi)$ is the pressure originating under the stamp; $x_0 = a$, where $2a$ is the size of the contact area; $f(x) = df_1(x)/dx$, where $f_1(x)$ is the shape of the contacting surface

$$A_0 = \pi^{-1} l_0 (l_4 - l_1), \qquad A_1 = \pi^{-1} l_0 (l_5 - l_2), \qquad A_2 = \pi^{-1} l_0 (l_5 - l_2) \times$$
$$\times (C^\circ + \ln \xi_0), \quad A_3 = \pi^{-1} l_0 (l_5 - l_2)(C^\circ + \ln \xi_0 + \xi_0)$$

Eq. (3.1) can be written as follows:

$$f(x) = \int_{-1}^{1} P(\xi) \left[\frac{A_0}{x-\xi} + A_1 \ln|x-\xi| + K^*(x-\xi) \right] d\xi$$

where the first member in the kernel of this integral equation is due to the elastic properties of the material, the second to the viscoelastic, and finally $K^*(x-\xi)$ is a regular function, also dependent on the viscoelastic properties of the medium.

Let us seek an approximate solution of the integral equation obtained.

Let us consider the first approximation. We will have

$$f(x) = \int_{-1}^{1} P(\xi)\left[\frac{A_0}{x-\xi} + A_1 \ln|x-\xi|\right] d\xi \tag{3.2}$$

It is easy to see that

$$\int_{-1}^{1} P(\xi)\ln|x-\xi|\,d\xi = \mathrm{Re}\left[\int_{-1}^{1} P(\xi)\ln(x-\xi)\,d\xi\right] \tag{3.3}$$

Integrating by parts, we obtain

$$\int_{-1}^{1} P(\xi)\ln(x-\xi)\,d\xi = i\pi\int_{-1}^{1} P(\xi)\,d\xi + \ln(1-x)\int_{-1}^{1} P(\xi)\,d\xi +$$
$$+\int_{-1}^{1}\left[\int_{-1}^{\xi} P(\xi)\,d\xi\right]\frac{d\xi}{x-\xi}$$

Then (3.2) becomes

$$\int_{-1}^{1}\left[A_0 P(\xi) + A_1\int_{-1}^{\xi} P(\xi)\,d\xi\right]\frac{d\xi}{x-\xi} = f(x) - A_1\ln(1-x)\int_{-1}^{1} P(\xi)\,d\xi \tag{3.4}$$

Let us introduce the following notation:

$$q(\xi) = A_0 P(\xi) + A_1\int_{-1}^{\xi} P(\xi)\,d\xi, \qquad F(x) = f(x) - A_1\ln(1-x)\int_{-1}^{1} P(\xi)\,d\xi \tag{3.5}$$

We therefore arrive at a Carleman equation of the first kind

$$\int_{-1}^{1} q(\xi)\frac{d\xi}{x-\xi} = F(x) \tag{3.6}$$

We seek the solution as some series

$$q(\xi) = \sum_{n=1}^{\infty} B_n\frac{T_n(\xi)}{\sqrt{1-\xi^2}} + \frac{B_0}{\sqrt{1-\xi^2}} \tag{3.7}$$

Here $T_n(\xi) = \cos(n \arccos \xi)$ are Chebyshev polynomials. Let us substitute (3.7) into (3.6), and let us transform the integral

$$\int_{-1}^{1}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}\,\frac{d\xi}{\xi-x} = \pi\sqrt{\frac{1-T_n^2(x)}{1-x^2}} = \frac{\pi}{n}T_n'(x)$$

Let us assume the function $F(x)$ defined by (3.5), can be expanded in derivatives of the Chebyshev polynomials in the considered interval $(-1, 1)$:

$$F(x)=\sum_{n=1}^{\infty}\varepsilon_n T_n'(x) \tag{3.8}$$

We then arrive at a system of algebraic equations to determine the coefficients B_n:

$$\sum_{n=1}^{\infty}\varepsilon_n T_n'(x)=-\sum_{n=1}^{\infty}B_n\frac{\pi}{n}T_n'(x)-B_0\int_{-1}^{1}\frac{1}{\sqrt{1-\xi^2}}\frac{d\xi}{\xi-x}, \qquad T_0(x)=1$$

Hence

$$B_n=-(n/\pi)\,\varepsilon_n \qquad (n=1,2,3,\ldots) \tag{3.9}$$

The constant B_0 is determined from the condition that the pressure acting on the stamp equals a given value

$$\frac{P}{E}=\int_{-1}^{1}P(\xi)\,d\xi \tag{3.10}$$

Let us now assume that

$$P(\xi)=r'(\xi) \tag{3.11}$$

Then, taking account of the relationship (3.5), we obtain

$$A_0 r'(\xi)+A_1 r(\xi)=q(\xi) \tag{3.12}$$

where $q(\xi)$ is a solution of (3.6). Hence

$$r(\xi)=\frac{1}{A_0}\exp\frac{-A_1\xi}{A_0}\int q(\xi)\exp\frac{A_1\xi}{A_0}\,d\xi \tag{3.13}$$

where the arbitrary constant is determined from the relationship (3.4). Taking the above into account, we write the final expression for the contact stress as

$$P(\xi)=\frac{1}{A_0}\left[-\sum_{n=1}^{\infty}\frac{n}{\pi}\varepsilon_n\frac{T_n(\xi)}{\sqrt{1-\xi^2}}+\frac{B_0}{\sqrt{1-\xi^2}}\right]- \tag{3.14}$$

$$-\frac{A_1}{A_0^2}\exp\frac{-A_1\xi}{A_0}\left[M+\int_{-1}^{\xi}\exp\frac{A_1\xi}{A_0}\left(-\sum_{n=1}^{\infty}\frac{n}{\pi}\varepsilon_n\frac{T_n(\xi)}{\sqrt{1-\xi^2}}+\frac{B_0}{\sqrt{1-\xi^2}}\right)d\xi\right]$$

We obtain $M=0$ from the relationship (3.4). Utilizing condition (3.10), we find

$$B_0=\frac{1}{\pi I_0(-A_1/A_0)}\left\{A_0\frac{P}{E}\exp\frac{A_1}{A_0}+\sum_{n=1}^{\infty}(-1)^n n\varepsilon_n I_n\left(-\frac{A_1}{A_0}\right)\right\} \tag{3.15}$$

Here $I_n(x)$ are Bessel functions of purely imaginary argument.
Let us examine the integrals in the relationship (3.14):

$$\int_{-1}^{\xi}\exp\frac{A_1\xi}{A_0}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}\,d\xi$$

By substituting $\theta=\arccos\xi$ and also decomposing the exponential function into power series, these integrals can be reduced to integrals of the form

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{A_1}{A_0}\right)^k\int_{\arccos\xi}^{\pi}\cos^k\theta\cos n\theta\,d\theta$$

for which recursion relations exist.

Let us find the second approximation for the solution of (3.1). We will have

$$f(x) = \int_{-1}^{1} P(\xi)\left[\frac{A_0}{x-\xi} + A_1 \ln|x-\xi| + A_2 + A_3(x-\xi) + A_1(x-\xi)\ln|x-\xi|\right]d\xi \qquad (3.16)$$

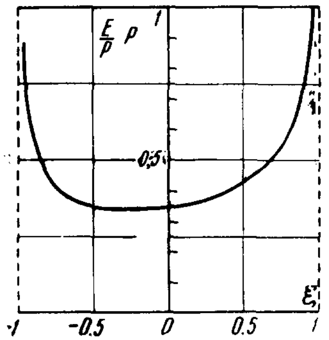

Fig. 2

Transforming (3.16) by the same method as for (3.2), we obtain

$$\int_{-1}^{1} q_1(\xi)\frac{d\xi}{x-\xi} = F_1(x) \qquad (3.17)$$

$$q_1(\xi) = A_0 P(\xi) + A_1 \int_{-1}^{\xi} P(\xi)\,d\xi + A_1 \int_{-1}^{\xi}\int_{-1}^{\xi} P(\xi)\,d\xi\,d\xi$$

$$F_1(x) = f(x) + C_0^*[A_1\ln(1-x) + A_2 - A_3(1-x) - A_1(1-x)\ln(1-x)] + C_1^*[A_1 + A_3 + A_1\ln(1-x)]$$

$$C_0^* = \int_{-1}^{1} P(\xi)\,d\xi, \qquad C_1^* = \int_{-1}^{1}\int_{-1}^{\xi} P(\xi)\,d\xi\,d\xi$$

Subsequent solutions are performed analogously to those presented above.

4. Let us consider an example. Let a rigid stamp with flat rectilinear base of width $2a$ move with constant velocity over the boundary of a viscoelastic half-plane. The half-plane material is polymethylmetacrylate.

In studying the mechanical properties of polymethylmetacrylate at high loading rates it has been established [6] that the dependence

$$\sigma = E\varepsilon + A\int_{-\infty}^{t} e^{-\frac{t-\tau}{\alpha}}\frac{d\varepsilon}{d\tau}\,d\tau$$

can be used, where E, A and α chosen from the condition of best agreement with experimental data are: $E = 8\times10^{10}$ dyne/cm², $A = 14.5\times10^{11}$ dyne/cm², $\alpha = 0.5\times10^{-6}$ sec.

It has also been indicated in [6] that volume creep is negligibly small for polymethylmetacrylate. The other constants are: Poisson coefficient $\nu = 0.36$; the density is 1.45g/cm³. We assume the velocity of stamp motion to be 300 m/sec.

The distribution of the pressure originating under the stamp is shown in Fig. 2.

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